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A theoretical study of the internal conversion with special application to magnetic dipole transitions

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With 4 figures in the text

SUMMARY

The internal conversion process has been analyzed by taking advantage of the simplifications in the mathematical formalism which are possible near the conversion threshold. The results are applied to $M1$ conversion in the K-shell. Corrections for various nuclear charge distributions can be easily made and a general insight into the nature of the problem can be obtained, allowing such predictions as the maximum deviation possible from the so-called normal values for the internal conversion coefficients that were first computed by Sliv and Band.

Introduction

In nuclear spectroscopy, the study of the internal conversion of nuclear gamma-rays supplies important information about the excited states of nuclei. During the last fifteen years there has been a rapid improvement in the facilities for making more precise calculations and accurate experimental determinations of the internal conversion coefficients. The development of fast electronic computers has made their accurate computation possible (Rose *et al.* 1951, Sliv and Band 1956 and 1958, Rose 1958) and the beta-spectrometric method for the experimental study of the internal conversion in complex spectra has been improved (Hultberg and Stockendal 1958).

With the publication of extensive tables for the K- and L-shells (Sliv and Band 1956 and 1958, Rose 1958) and the M-shell (Rose 1958), the experimentalist has obtained a fairly easy access to the internal conversion coefficients for such values of the gamma-ray energy and the multipolarity which are of interest in practice, at least for the K- and L-shells.

In the earlier calculations of Rose, the nuclear dimension were neglected. This is generally a very good approximation but, in spite of the small dimensions of the nucleus as compared with the atomic dimensions and the wave length of the nuclear gamma-ray, the nucleus can have a considerable influence on the internal conversion rate. This circumstance was first realized by Sliv. (Sliv 1951, Sliv and Listengarten 1952.) In 1956, Sliv and Band published comprehensive tables of conversion coefficients. These computations were based on the assumption that the displacement of nuclear matter takes place on the nuclear surface only and this means that there is no interaction between nuclear matter and the atomic shell electrons when the latter are inside the nucleus. Mathematically this corresponds to cutting off the radial part of the transition matrix elements inside the nucleus. The presence of the nucleus makes it necessary to correct the electron wave functions. The values for the internal conversion coefficients obtained using this model differ by as much as 50 per cent from the point-nucleus values which were computed by Rose. These large differences are mainly caused by the removal of the singularities in the radiation field as well as in the electron wave functions for certain states as a consequence of the finite nuclear size. As soon as the singularities are removed, the internal conversion coefficients are rather insensitive for details in the nuclear structure and thus the simple model used by Sliv and Band should agree well with real conditions. Green and Rose (1958) and Rose (1958) have recently recalculated the internal conversion coefficients taking the finite size of the nucleus into account, using a nuclear model differing from that of Sliv and Band. The so-called "no-penetration" model of Rose confines the radiation sources to the nuclear center whereas Sliv and Band put them on the nuclear surface. In the no-penetration model, as in any physically reasonable model, the singularities at the origin are removed and thus we expect a general agreement between the results obtained from the surface model of Sliv and Band and from the no-penetration model as well as from any other physically reasonable model. This statement is justified by the investigations reported in the present paper. As a rule then, the contribution to the interaction between shell electrons and nuclear matter which comes from the intra-nuclear region is negligible as compared with the contribution from the extra-nuclear region. Provided that this is the case, the statement that the internal conversion coefficients are independent of the nuclear structure is still valid within a few per cent. The contribution to the conversion matrix element from the extra-nuclear region is, however, proportional to the gamma-ray field and is thus very small when the gamma-ray emission is very weak. This contribution may then be of the same order of magnitude as the intra-nuclear contribution. In such cases, the internal conversion coefficients may depend considerably on the nuclear structure. Kramer (1956, 1957) has given the expressions for the conversion coefficients for a finite nucleus and Church and Weneser (1956) pointed out the possibility of large deviations of the conversion coefficients from the Sliv values and investigated some slow magnetic dipole transitions. Similar investigations have also been performed by Reiner (1958) and Nilsson and Rasmussen (1958). The latter authors investigated electric dipole transitions.

In spite of the comprehensive tables for the conversion coefficients that are available, a theoretical investigation should be interesting both from the theoretical and the practical points of view since such an investigation should, among other things, simplify the application of corrections for disturbing effects like the influence of the finite nuclear size. In the present paper the conversion process for energies near the conversion threshold is studied and the results are applied to magnetic dipole conversion in the K-shell. The results obtained for the very low energies near the threshold hold also for energies which are as high as ten times the binding energy of the bound electron, and, for the heavy nuclei, extend well into the energy region of physical interest.

Internal conversion matrix elements

Since the detailed calculations leading to the expression for the internal conversion coefficients in terms of radial integrals may be found elsewhere, we will not attempt here a derivation of these expressions, but will adopt the results obtained by Rose *et al.* (1951) and Rose (1955). These results do not take the finite nuclear size into account. Such effects will be considered in the following.

In this paper, we shall confine ourselves mainly to conversion in the K-shell for magnetic transitions. For such conversion coefficients, the following expression has been given by Rose

$$\beta_L^K = \frac{2\pi\alpha k}{2L+1}\left[L\,|R_{\varkappa'=L+1}(L)|^2 + (L+1)\,|R_{\varkappa'=-L}(L)|^2\right], \tag{1}$$

where L is the multipole order of the gamma-ray, α the fine-structure constant, k is the gamma-ray energy in units of mc^2 and where $R_{\varkappa'}(L)$ contains the radial integrals of the transition matrix elements.

The expressions for the radial integrals become simpler and are more easily studied if a representation of the electron wave functions, introduced by one of us (Olsson 1959 a), is used. These expressions are

$$\begin{aligned} \chi_1 &= \alpha Z t^{\gamma}\left[\Phi(\gamma, E, t) + \frac{E\varkappa+\gamma}{\varkappa+\gamma}\,\frac{t}{(2\gamma)_2}\,\Phi(\gamma+1, E, t)\right] \\ \chi_2 &= (\varkappa-\gamma)\, t^{\gamma}\left[\Phi(\gamma, E, t) + \frac{E\varkappa+\gamma}{\varkappa-\gamma}\,\frac{t}{(2\gamma)_2}\,\Phi(\gamma+1, E, t)\right], \end{aligned} \tag{2}$$

where $$(2\gamma)_2 = 2\gamma(2\gamma+1).$$

Furthermore, Z is the nuclear charge number, E is the electron energy, including its rest mass, in units of mc^2, $t = 2\alpha Z\, r$ with r measured in units of $\hbar/mc$ and finally $\gamma = \pm\sqrt{\varkappa^2 - \alpha^2 Z^2}$ where $\varkappa = \pm 1, \pm 2, \pm 3, \ldots$. Both signs for γ are possible and the functions thus obtained are linearly independent solutions of the radial part of the Dirac equation:

$$\begin{aligned} \frac{df(r)}{dr} &= \frac{\varkappa-1}{r} f(r) - \left(E - 1 + \frac{\alpha Z}{r}\right) g(r) \\ \frac{dg(r)}{dr} &= \left(E + 1 + \frac{\alpha Z}{r}\right) f(r) - \frac{\varkappa+1}{r} g(r), \end{aligned} \tag{3}$$

where $$\chi_1 = r f(r),\ \chi_2 = r g(r).$$

The functions $\Phi(\gamma, E, t)$ are defined by

$$\Phi(\gamma, E, t) = e^{-ipr}\,{}_1F_1(\gamma + i\eta;\, 2\gamma;\, 2ipr) \tag{4}$$

where $$p = \sqrt{E^2 - 1} \quad \text{and} \quad \eta = \frac{E\alpha Z}{p}.$$

For a point-nucleus, we must choose the positive value of γ, otherwise the wave functions do not give finite electron densities at the origin. Introducing the wave functions (2) into the expressions for $R_{\varkappa'}(L)$ as they are given by Rose, we obtain

$$R_{\varkappa'}(L)=N_i N_f \int_0^\infty h_L(kr)\,[\chi_1^{(i)}\chi_2^{(f)}+\chi_1^{(f)}\chi_2^{(i)}]\,dr \tag{5}$$

where the letters i and f refer to initial (bound) states and final (continuum) states, respectively. R depends on $\varkappa'$ of the final state. Angular selection rules forbid transitions other than those for which $\varkappa'=-L$ or $+(L+1)$. The quantities N are normalization constants of the electron wave functions and $h_L(kr)$ is a spherical Hankel function of the first kind

$$h_L(kr)=\sqrt{\frac{\pi}{2kr}}\,H^{(1)}_{L+\frac{1}{2}}(kr)=(-i)^{L+1}\frac{e^{ikr}}{kr}\,{}_2F_0\left(-L,\,L+1,\,\frac{1}{2ikr}\right). \tag{6}$$

In agreement with common notations, the ${}_2F_0$-function is defined by

$${}_2F_0\left(-L,\,L+1;\,\frac{1}{2ikr}\right)=\sum_{n=0}^{\infty}\frac{(-L)_n\,(L+1)_n}{n!}\left(\frac{1}{2ikr}\right)^n \tag{7}$$

where a symbol such as $(a)_n$ means

$$(a)_n=a(a+1)(a+2)\,\dots\,(a+n-1),\quad a_0=1.$$

The expansion (7) breaks off when $-L+n-1=0$ and is thus a polynomial of degree L in $\frac{1}{2ikr}$.

The spherical Hankel function can be expressed in terms of spherical Bessel functions.

$$h_L(kr)=j_L(kr)-i(-1)^L j_{-L}(kr) \tag{8}$$

where

$$j_L(kr)=\sqrt{\frac{\pi}{2kr}}\,J_{L+\frac{1}{2}}(kr),$$

$$j_{-L}(kr)=\sqrt{\frac{\pi}{2kr}}\,J_{-L-\frac{1}{2}}(kr).$$

This splits the Hankel function into its real and imaginary parts. Inserting (8) into (5) we obtain

$$R_{\varkappa'}(L)=N_i N_f\int_0^\infty j_L(kr)\,[\chi_1^{(i)}\chi_2^{(f)}+\chi_1^{(f)}\chi_2^{(i)}]\,dr-$$

$$-i(-1)^L N_i N_f\int_0^\infty j_{-L}(kr)\,[\chi_1^{(i)}\chi_2^{(f)}+\chi_1^{(f)}\chi_2^{(i)}]\,dr. \tag{9}$$

The constants N_i and N_f, as well as both integrals, are all real. As is well-known, the Bessel function can be expressed in terms of confluent hypergeometric functions:

$$J_\nu(x) = \frac{1}{\Gamma(1+\nu)}\left(\frac{x}{2}\right)^{\nu} e^{-ix}\,{}_1F_1(\nu+\tfrac{1}{2};\,2\nu+1;\,2ix).$$

We thus obtain

$$\begin{aligned} j_L(kr) &= \tfrac{1}{2}\left(\frac{kr}{2}\right)^{L}\frac{\sqrt{\pi}}{\Gamma(L+\frac{3}{2})}\,e^{-ikr}\,{}_1F_1(L+1;\,2L+2;\,2ikr),\\ j_{-L}(kr) &= \frac{(-1)^L}{2}\left(\frac{2}{kr}\right)^{L+1}\frac{\Gamma(L+\frac{1}{2})}{\sqrt{\pi}}\,e^{-ikr}\,{}_1F_1(-L,\,-2L;\,2ikr). \end{aligned} \tag{10}$$

The last relation is not true if L is a non-negative integer but it can still be used provided that the integrals in which the function occurs in the integrand are defined also for the exceptional values of L. If the function itself is needed we can use

$$j_{-L}(kr) = \frac{(-1)^L}{2}\left(\frac{2}{kr}\right)^{L+1}\frac{\Gamma(L+\frac{1}{2})}{\sqrt{\pi}}\,{}_0F_1\left(\tfrac{1}{2}-L;\,-\frac{k^2r^2}{4}\right)$$

but we prefer to use (10) here.

The normalized wave functions for the K-shell are

$$\begin{aligned} N_i\chi_1 &= -\sqrt{\frac{\alpha Z(1-\gamma)}{\Gamma(1+2\gamma)}}\,t^{\gamma}e^{\frac{t}{2}},\\ N_i\chi_2 &= \sqrt{\frac{\alpha Z(1+\gamma)}{\Gamma(1+2\gamma)}}\,t^{\gamma}e^{-\frac{t}{2}}, \end{aligned} \tag{11}$$

were $\gamma=\sqrt{1-\alpha^2Z^2}$.

We can then express $R_{\varkappa'}(L)$ in the following manner:

$$\begin{aligned} R_{\varkappa'}(L) = N_f\sqrt{\frac{\alpha Z(1-\gamma)}{\Gamma(1+2\gamma)}}\Big[&(1+\gamma+\gamma'-\varkappa')\,J_1(L,|\varkappa'|)+\\ &+\frac{E'\varkappa'+\gamma'}{\varkappa'+\gamma'}\,\frac{1}{(2\gamma')_2}\,(1+\gamma-\gamma'-\varkappa')\,J_2(L,|\varkappa'|)\Big],\quad \gamma'=\sqrt{\varkappa'^2-\alpha^2Z^2}. \end{aligned} \tag{12}$$

Here all primed quantities refer to the final state that is uniquely determined by $\varkappa'$ and its energy E'. The integral $J_1(L,|\varkappa'|)$ is

$$J_1(L,|\varkappa'|) = \int_0^\infty e^{-\frac{t}{2}}\,t^{\gamma+\gamma'}\,h_L(kr)\,e^{-ipr}\,{}_1F_1(\gamma'+i\eta;\,2\gamma';\,2ipr)\,dr \tag{13}$$

and $J_2(L,|\varkappa'|)$ is obtained by increasing γ' in $J_1(L,|\varkappa'|)$ by one unit.

Inserting (10) into (13) we obtain for the real and imaginary parts:

$$\mathrm{Re}\, J_1(L, |\varkappa'|) = \frac{2^{\gamma+\gamma'} k^L \sqrt{\pi}}{(2\alpha Z)^{L+1}\Gamma(L+\frac{3}{2})} \int_0^\infty e^{-s} s^{\gamma+\gamma'+L} e^{-ik's} {}_1F_1(L+1; 2L+2; 2ik's) \cdot e^{-ip's} {}_1F_1(\gamma'+i\eta; 2\gamma'; 2ip's)\, ds. \quad (14)$$

$$\mathrm{Im}\, J_1(L, |\varkappa'|) = -\frac{2^{\gamma+\gamma'} (2\alpha Z)^L \Gamma(L+\frac{1}{2})}{k^{L+1}\sqrt{\pi}} \int_0^\infty e^{-s} s^{\gamma+\gamma'-L-1} e^{-ik's} {}_1F_1(-L; -2L; 2ik's) \cdot e^{-ip's} {}_1F_1(\gamma'+i\eta; 2\gamma'; 2ip's)\, ds.$$

Here, we have introduced $s=\frac{t}{2}$, $p'=\frac{p}{\alpha Z}$ and $k'=\frac{k}{\alpha Z}$.

The two integrals (14) may be expressed in terms of the Appell-function $F_2(a, b, b'; c, c'; x, y)$ (Erdélyi 1936).

$$\mathrm{Re}\, J_1(L, |\varkappa'|) = \frac{2^{\gamma+\gamma'} k^L \sqrt{\pi}}{(2\alpha Z)^{L+1}\Gamma(L+\frac{3}{2})} \frac{\Gamma(\gamma+\gamma'+L+1)}{[1+i(p'+k')]^{\gamma+\gamma'+L+1}} \cdot F_2\left(\gamma+\gamma'+L+1, L+1, \gamma'+i\eta; 2L+2, 2\gamma'; \frac{2ik'}{1+i(k'+p')}, \frac{2ip'}{1+i(k'+p')}\right). \quad (15)$$

$$\mathrm{Im}\, J_1(L, |\varkappa'|) = -\frac{2^{\gamma+\gamma'} (2\alpha Z)^L \Gamma(L+\frac{1}{2})}{k^{L+1}\sqrt{\pi}} \frac{\Gamma(\gamma+\gamma'-L)}{[1+i(k'+p')]^{\gamma+\gamma'-L}} \cdot F_2\left(\gamma+\gamma'-L, -L, \gamma'+i\eta; -2L, 2\gamma'; \frac{2ik'}{1+i(k'+p')}, \frac{2ip'}{1+i(k'+p')}\right).$$

In the neighbourhood of the origin, the F_2-function is defined by

$$F_2(a, b, b'; c, c'; x, y) = \sum_{\substack{m=0\\n=0}}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(c)_m (c')_n m!\, n!}.$$

Convergent expansions for integrals of the type occurring here have been given by Olsson (1958 b). The expression for the imaginary part fails for $L=0, 1, 2, 3, \ldots$. We therefore introduce a transformation given by Erdélyi (1948).

$$F_2(a, b, b'; 2b, c'; x, y) = \left(1-\frac{x}{2}\right)^{-a} H_4\left(a, b'; b+\tfrac{1}{2}, c'; \frac{x^2}{4(2-x)^2}, \frac{2y}{2-x}\right) \quad (16)$$

where
$$H_4(a, b; c, d; x, y) = \sum_{\substack{m=0\\n=0}}^{\infty} \frac{(a)_{2m+n}(b)_n x^m y^n}{(c)_m (d)_n m!\, n!}.$$

Due to the introduction of the H_4-function the J-integrals will be defined also for integral L-values:

$$\operatorname{Re} k^{L+1} J_1(L, |\varkappa'|) = \frac{2^{\gamma+\gamma'} k^{2L+1}}{(2\alpha Z)^{L+1}} \frac{\sqrt{\pi}}{\Gamma(L+\frac{3}{2})} \frac{\Gamma(\gamma+\gamma'+L+1)}{(1+ip')^{\gamma+\gamma'+L+1}} \cdot$$

$$\cdot H_4(\gamma+\gamma'+L+1, \gamma'+i\eta; L+\tfrac{3}{2}, 2\gamma'; x, y), \quad (17)$$

$$\operatorname{Im} k^{L+1} J_1(L, |\varkappa'|) = -i \frac{2^{\gamma+\gamma'} (2\alpha Z)^L \Gamma(L+\frac{1}{2})}{\sqrt{\pi}} \frac{\Gamma(\gamma+\gamma'-L)}{(1+ip')^{\gamma+\gamma'-L}} \cdot$$

$$\cdot H_4(\gamma+\gamma'-L, \gamma'+i\eta; \tfrac{1}{2}-L, 2\gamma'; x, y),$$

where
$$x = \left[\frac{ik'}{2(1+ip')}\right]^2, \quad y = \frac{2ip'}{1+ip'}.$$

By confluence in (16) we obtain

$$\lim_{\varepsilon\to 0} F_2\left(a, b, \frac{1}{\varepsilon}; 2b, c'; x, \varepsilon y\right) = \Psi_1(a, b; 2b, c'; x, y) =$$

$$= \lim_{\varepsilon\to 0} \left(1-\frac{x}{2}\right)^{-a} H_4\left(a, \frac{1}{\varepsilon}; b+\tfrac{1}{2}, c'; \frac{x^2}{4(2-x)^2}, \frac{2\varepsilon y}{2-x}\right) = \quad (18)$$

$$= \left(1-\frac{x}{2}\right)^{-a} \mathrm{H}_7\left(a; b+\tfrac{1}{2}, c'; \frac{x^2}{4(2-x)^2}, \frac{2y}{2-x}\right),$$

where
$$\Psi_1(a, b; c, c'; x, y) = \sum_{\substack{m=0\\n=0}}^{\infty} \frac{(a)_{m+n}(b)_m x^m y^n}{(c)_m (c')_n \, m! \, n!}$$

and
$$\mathrm{H}_7(a; b, c; x, y) = \sum_{\substack{m=0\\n=0}}^{\infty} \frac{(a)_{2m+n} x^m y^n}{(b)_m (c)_n \, m! \, n!}.$$

These results are easily obtained by using the limit

$$\lim_{\varepsilon\to 0} \left(a+\frac{b}{\varepsilon}\right)_n (\varepsilon x)^n = (bx)^n$$

in each term in the expansions of the functions in question. The parameter $\eta = \frac{E\alpha Z}{p}$ tends to infinity when $p \to 0$ and the H_4-functions in (17) will therefore have its confluent function H_7 as a limit when $p = 0$. Conservation of energy requires

$$k = E - \gamma$$

and the gamma-ray energy for $p = 0$ is then

$$k_0 = 1 - \gamma$$

which is the threshold energy for conversion.

Internal conversion matrix elements near the conversion threshold

From the result (18) of the previous section the threshold values of the J-integrals can be obtained directly:

$$\operatorname{Re} k_0^{L+1} J_1(L, |\varkappa'|) = 2^{\gamma+\gamma'} (2\alpha Z)^L \left(\frac{1-\gamma}{2\alpha Z}\right)^{2L+1} \frac{\sqrt{\pi}}{\Gamma(L+\frac{3}{2})} \Gamma(\gamma+\gamma'+L+1)\cdot$$

$$\cdot \mathrm{H}_7(\gamma+\gamma'+L+1;\, L+\tfrac{3}{2};\, 2\gamma';\, x_0, y_0).$$

$$\operatorname{Im} k_0^{L+1} J_1(L, |\varkappa'|) = -2^{\gamma+\gamma'} (2\alpha Z)^L \frac{\Gamma(L+\frac{1}{2})}{\sqrt{\pi}} \Gamma(\gamma+\gamma'-L)\cdot$$

$$\cdot \mathrm{H}_7(\gamma+\gamma'-L;\, \tfrac{1}{2}-L,\, 2\gamma';\, x_0, y_0), \quad (19)$$

where $$x_0 = -\frac{1-\gamma}{4(1+\gamma)}, \quad y_0 = -2.$$

All transitions, the electric as well as the magnetic, can be expressed in terms of the $J(L, |\varkappa'|)$-integrals for which a convergent expansion can be obtained (Olsson 1959 c). This expansion is convergent for all values of the parameters, which are of interest in connection with internal conversion and has been used to some extent for checking results which have been obtained from simpler, approximate formulas. The complicated mathematical structure of the problem seems to make it impractical to try to obtain simple expressions for the conversion coefficients unless the gamma-ray energy is very low or very high. We will here focus our interest on the low energies. This simplifies the expressions considerably but the conclusions which can be drawn from the simpler formulas are in many cases of a very restricted validity since they are in general true only for very small energies. In the present work, we will investigate the internal conversion process in magnetic dipole transitions by taking advantage of the simplifications that are possible for low energies.

Such an investigation is most interesting for the heavier nuclei for which also the validity of the simple expressions, which will be derived for the conversion coefficients, extend well into the energy region of physical interest. The energy dependence of the J-integrals is very complicated due to the fact that the energy is contained in the parameter η as well as in the arguments of the functions in the integrands and many terms seem to be needed in the convergent expansions of the integrals, even for very small p-values. However, if the validity of the expansion is restricted to very small p-values, an expansion in powers of p can be used and the result obtained using only a few terms may hold good far beyond the value of p where the convergence ceases. The possibilities and short-comings of an analysis based solely on a low energy investigation can be fairly well predicted from existing data. Besides, the information obtained can be completed by an investigation of the high energy limit.

In order to obtain an approximate formula for the conversion coefficients on the very threshold, we observe that x_0 in (19) is small since $\frac{1-\gamma}{4(1+\gamma)} < 0.04$ for all nuclei. The expansion given for the H_7-function converges when $|x| < \frac{1}{4}$ for all values of y and the expansion of the H_7-function can thus be expected to be rapidly convergent when summed over x. The summation over y is easily performed. We put

$$\mathrm{H}_7(a; b, c; x, y) = \sum_{m,n} \frac{(a)_{2m}(a+2m)_n x^m y^n}{(b)_m (c)_n m!\, n!} =$$

$$= \sum_m \frac{(a)_{2m} x^m}{(b)_m m!} \sum_n \frac{(a+2m)_n y^n}{(c)_n n!} = \sum_m \frac{(a)_{2m} x^m}{(b)_m m!} \, {}_1F_1(a+2m; c; y).$$

For the small values of x obtained above, we simply put

$$\mathrm{H}_7(a; b, c; x, y) \approx {}_1F_1(a; c; y), \quad |x| \ll \tfrac{1}{4}$$

and obtain

$$\mathrm{Re}\, k_0^{L+1} J_1(L, |\varkappa'|) \approx 2^{\gamma+\gamma'} (2\alpha Z)^L \left(\frac{1-\gamma}{2\alpha Z}\right)^{2L+1} \frac{\Gamma(\gamma+\gamma'+L+1)\sqrt{\pi}}{\Gamma(L+\frac{3}{2})} \cdot$$

$$\cdot {}_1F_1(\gamma+\gamma'+L+1; 2\gamma'; -2).$$

$$\mathrm{Im}\, k_0^{L+1} J_1(L, |\varkappa'|) \approx -2^{\gamma+\gamma'} (2\alpha Z)^L \frac{\Gamma(\gamma+\gamma'-L)\,\Gamma(L+\frac{1}{2})}{\sqrt{\pi}} \, {}_1F_1(\gamma+\gamma'-L; 2\gamma'; -2). \tag{20}$$

In these expressions we have obviously neglected the binding energy of the converted electron. The most important contribution to the magnetic dipole conversion is the one where the converted electron is expelled in an $s_{\frac{1}{2}}$-state. We then have $\varkappa' = -1$ and thus $\gamma' = \gamma$ and hence we obtain

$$\begin{aligned} \mathrm{Re}\, k_0^2 J_1(1, 1) &\approx -\frac{2}{3} e^{-2} 2^{2\gamma} \Gamma(2\gamma) \frac{\gamma(3-2\gamma)(1-\gamma)^2}{1+\gamma}, \\ \mathrm{Re}\, k_0^2 J_2(1, 1) &\approx +\frac{2}{3} e^{-2} 2^{2\gamma} \Gamma(2\gamma)\, 2\gamma\, (2\gamma)_2 \frac{(1-\gamma)^2}{1+\gamma}. \end{aligned} \tag{21}$$

It can be seen from (12) that the factor in front of the $J_1(1, 1)$ integral in $R_{-1}(1)$ is $(1+\gamma)(2\gamma)_2$ times larger than that in front of the $J_2(1, 1)$ integral. The real part of the latter integral is $2(2\gamma)_2/(3-2\gamma)$ times larger than the real part of the former and is of opposite sign and hence there is a very good cancellation of the real terms. For the imaginary parts we obtain

$$\begin{aligned} \mathrm{Im}\, k_0^2 J_1(1, 1) &\approx -\alpha Z e^{-2} 2^{2\gamma} \Gamma(2\gamma-1)\, {}_1F_1(1; 2\gamma; 2), \\ \mathrm{Im}\, k_0^2 J_2(1, 1) &\approx -\alpha Z e^{-2} 2^{2\gamma+1} \Gamma(2\gamma)\, {}_1F_1(2, 2\gamma+2, 2). \end{aligned} \tag{22}$$

The terms which are omitted in (22) contain only ${}_1F_1$-functions that are polynomials. In the expression for the $J_1(1, 1)$ integral the error is of the order of

$(1-\gamma)^2$ since the coefficient of $(1-\gamma)$ is zero. For light nuclei, the value of the imaginary part of the $J_1(1,1)$ integral is about half the value of the imaginary part of the $J_2(1,1)$ integral whereas, for the heaviest nuclei, the ratio is about 2. The two integrals are equal for a Z value of about 84. Furthermore we easily obtain

$$\left|\frac{\mathrm{Re}\; J_1(1,1)}{\mathrm{Im}\; J_1(1,1)}\right| < 0.005, \qquad \left|\frac{\mathrm{Re}\; J_2(1,1)}{\mathrm{Im}\; J_2(1,1)}\right| < 0.04.$$

For light nuclei, these ratios are much smaller. Due to the above-mentioned cancellation of the real terms, the real part of the matrix element is much smaller than the above ratios would indicate. In fact, we have

$$\left|\frac{\mathrm{Re}\; R_{-1}(1)}{\mathrm{Im}\; R_{-1}(1)}\right| < 0.001.$$

We can therefore safely neglect the real part of the matrix element.

We now consider transitions to the $d_{3/2}$-states. A rough numerical estimate gives

$$\left|\frac{R_{-1}(1)}{R_2(1)}\right|^2 > 10^4$$

for all Z-values. Thus the contributions from the transitions to the $d_{3/2}$-states can also be neglected for the very low energies considered here.

With an accuracy of 10^{-4}, we can thus write at low energies:

$$\beta_1^K = \frac{8\pi\alpha k}{3}\left| N_f \frac{\sqrt{\alpha Z(1-\gamma)}}{\sqrt{\Gamma(1+2\gamma)}}\left[(1+\gamma)\, J_1(1,1) + \frac{E-\gamma}{1-\gamma}\,\frac{1}{(2\gamma)_2}\, J_2(1,1)\right]\right|^2. \tag{23}$$

For the normalization constant N_f, we take the expression (Olsson 1959 a)

$$N_f = \left(\frac{p}{\alpha Z}\right)^{\gamma'} \sqrt{\frac{E'\varkappa' - \gamma'}{\varkappa' - \gamma'}}\; \frac{|\Gamma(\gamma' + i\eta)|\, e^{\frac{\pi}{2}\eta}}{\sqrt{2\pi p}\,\Gamma(1+2\gamma)}. \tag{24}$$

By means of asymptotic expansions of the Γ-function, we obtain, for small p-values

$$N_f = \frac{1}{\sqrt{\alpha Z}\,\Gamma(1+2\gamma')}\left[1 + \frac{6\varkappa^2\gamma' + 3\varkappa\gamma' + \gamma' - 4\gamma'^3}{12(\varkappa^2 - \gamma'^2)}\, p^2 + \cdots\right]. \tag{25}$$

In order to obtain in addition the values of the J-integrals in the vicinity of the conversion threshold, we calculate their derivatives with respect to the energy neglecting the real parts. We obtain, using (14),

$$\frac{\partial}{\partial E} k^{L+1} \operatorname{Im} \; J_1(1,1) = -\frac{2^{\gamma+\gamma'}(2\alpha Z)^L \Gamma(L+{}^1/_2)}{\sqrt{\pi}} \cdot$$

$$\cdot \left[\int_0^\infty e^{-s} s^{\gamma+\gamma'-L-1} e^{-ip's} {}_1F_1(\gamma'+i\eta;\, 2\gamma';\, 2ip's) \frac{\partial}{\partial E} {}_0F_1\left(\tfrac{1}{2}-L;\; -\frac{k^2 s^2}{4\alpha^2 Z^2}\right) ds + \right. \tag{26}$$

$$\left. + \int_0^\infty e^{-s} s^{\gamma+\gamma'-L-1} {}_0F_1\left(\tfrac{1}{2}-L;\; -\frac{k^2 s^2}{4\alpha^2 Z^2}\right) \frac{\partial}{\partial E} e^{-ip's} {}_1F_1(\gamma'+i\eta;\, 2\gamma';\, 2ip's)\, ds \right].$$

We first calculate the derivative of the Bessel function at the threshold

$$\left[\frac{\partial}{\partial E} {}_0F_1\left(\tfrac{1}{2}-L;\; -\frac{k^2 s^2}{4\alpha^2 Z^2}\right)\right]_{E=1} = \frac{s^2}{(2L-1)(1+\gamma)} {}_0F_1\left(\tfrac{3}{2}-L;\; -\frac{1-\gamma}{4(1+\gamma)} s^2\right). \tag{27}$$

In order to obtain the derivative of the Coulomb wave functions, we use an expansion of these functions in terms of Bessel functions (Tricomi 1949, Olsson 1959 a)

$$e^{-ip's} {}_1F_1(\gamma'+i\eta;\, 2\gamma';\, 2ip's) = \sum_{n=0}^{\infty} \frac{a_n}{(2\gamma')_n} (2s)^n {}_0F_1(2\gamma'+n;\, -2Es) \tag{28}$$

where the coefficients can be obtained from the recursion relation

$$(n+1)\, a_{n+1} = -\frac{p^2}{4\alpha^2 Z^2} [(n+2\gamma'-1)\, a_{n-1} - E\, a_{n-2}],$$

$$a_0 = 1, \;\; a_1 = 0 \quad \text{etc.}$$

Since the coefficient a_4 is of order of p^4, only the four first terms have to be considered when carrying out the differentiation. The result is

$$\left[\frac{\partial}{\partial E} e^{-ip's} {}_1F_1(\gamma'+i\eta;\, 2\gamma';\, 2ip's)\right]_{E=1} =$$

$$= -\frac{s}{\gamma'} {}_0F_1(2\gamma'+1,\, -2s) - \frac{s^2}{\alpha^2 Z^2 (2\gamma'+1)} {}_0F_1(2\gamma'+2;\, -2s) + \tag{29}$$

$$+ \frac{4 s^3}{3\alpha^2 Z^2 (2\gamma')_3} {}_0F_1(2\gamma'+3;\, -2s).$$

The derivatives calculated are inserted into (26). We observe that the integrals occurring will be of the same type as those which could be expressed in terms of H_7-functions. After suitable changes of the parameters we obtain

$$\left[\frac{\partial}{\partial E}\,\mathrm{Im}\,k^{L+1}J_1(1,1)\right]_{E=1} = -\frac{2^{\gamma+\gamma'}(2\alpha Z)^L\,\Gamma(L+\frac{1}{2})}{\sqrt{\pi}}\cdot$$
$$\cdot\left[\frac{\Gamma(\gamma+\gamma'-L+2)}{(2L-1)(1+\gamma)}\,\mathrm{H}_7(\gamma+\gamma'-L+2;\tfrac{3}{2}-L,\,2\gamma';\,x_0,y_0)-\right.$$
$$-\frac{\Gamma(\gamma+\gamma'-L+1)}{\gamma'}\,\mathrm{H}_7(\gamma+\gamma'-L+1;\tfrac{1}{2}-L;\,2\gamma'+1;\,x_0,y_0)- \tag{30}$$
$$-\frac{\Gamma(\gamma+\gamma'-L+2)}{\alpha^2Z^2(2\gamma'+1)}\,\mathrm{H}_7(\gamma+\gamma'-L+2;\tfrac{1}{2}-L,\,2\gamma'+2;\,x_0,y_0)+$$
$$\left.+\frac{4\Gamma(\gamma+\gamma'-L+3)}{3\alpha^2Z^2(2\gamma')_3}\,\mathrm{H}_7(\gamma+\gamma'-L+3;\tfrac{1}{2}-L,\,2\gamma'+3;\,x_0,y_0)\right]$$

where
$$x_0=-\frac{1-\gamma}{4(1+\gamma)},\qquad y_0=-2$$

as in (19). An approximate expression for the derivatives can be obtained by replacing the H_7-functions by the first term in their expansions as was done in (20) and using recursion relations for the ${}_1F_1$-functions to simplify the expression obtained.

With the aid of the results derived above, it is a matter of straightforward calculation to derive the coefficients in an expansion of the form

$$k^3\beta_1^K = A_E+(E-1)\,B+\cdots. \tag{31}$$

We obtain
$$A_E=(k^3\beta)_{E=1},$$
$$B=2A_E\left[\frac{\partial}{\partial E}\ln|k^2R_{-1}(1)|\right]_{E=1}$$

where

$$\left[\frac{\partial}{\partial E}\ln|k^2R_{-1}(1)|\right]_{E=1}=\left[\frac{\partial}{\partial E}\ln N_f\right]_{E=1}+$$
$$+\frac{(1+\gamma)\left[\frac{\partial}{\partial E}k^2J_1(1,1)\right]_{E=1}+\frac{1}{(2\gamma)_2}\left[\frac{\partial}{\partial E}k^2J_2(1,1)\right]_{E=1}+\frac{1}{(1-\gamma)(2\gamma)_2}k_0^2J_2(1,1)}{(1+\gamma)k_0^2J_1(1,1)+\frac{1}{(2\gamma)_2}k_0^2J_2(1,1)}.$$

For the derivatives we take the imaginary part (30).

The expansion (31) is convergent for small values of $E-1$ only. Nevertheless the simple expression

$$\beta_1^K=\frac{A+kB}{k^3} \tag{32}$$

gives correct values of the magnetic dipole conversion coefficients for the heavier nuclei even for gamma-energies as high as 0,7 MeV. Since it is common to

Table 1. The coefficients A and B.

Z	A	B	Z	A	B	T
30	0.000326	0.000244	64	0.0126	0.00927	4.10
31	0.000375	0.000281	65	0.0138	0.0102	4.19
32	0.000430	0.000323	66	0.0151	0.0111	4.27
33	0.000492	0.000369	67	0.0166	0.0122	4.36
34	0.000561	0.000420	68	0.0181	0.0133	4.46
35	0.000638	0.000477	69	0.0199	0.0146	4.55
36	0.000723	0.000541	70	0.0218	0.0160	4.65
37	0.000817	0.000611	71	0.0239	0.0175	4.75
38	0.000921	0.000688	72	0.0262	0.0192	4.85
39	0.00104	0.000774	73	0.0287	0.0211	4.96
40	0.00116	0.000868	74	0.0315	0.0231	5.07
41	0.00130	0.000971	75	0.0345	0.0254	5.18
42	0.00146	0.00109	76	0.0379	0.0278	5.30
43	0.00162	0.00121	77	0.0417	0.0306	5.42
44	0.00181	0.00135	78	0.0458	0.0336	5.55
45	0.00201	0.00150	79	0.0504	0.0370	5.68
46	0.00224	0.00166	80	0.0555	0.0407	5.82
47	0.00248	0.00184	81	0.0612	0.0449	5.97
48	0.00275	0.00204	82	0.0675	0.0497	6.12
49	0.00304	0.00226	83	0.0746	0.0547	6.28
50	0.00336	0.00249	84	0.0825	0.0606	6.44
51	0.00371	0.00275	85	0.0914	0.0671	6.62
52	0.00409	0.00303	86	0.101	0.0744	6.80
53	0.00451	0.00334	87	0.113	0.0826	7.00
54	0.00496	0.00367	88	0.125	0.0919	7.20
55	0.00546	0.00404	89	0.140	0.102	7.42
56	0.00600	0.00444	90	0.156	0.114	7.66
57	0.00659	0.00487	91	0.175	0.128	7.91
58	0.00724	0.00535	92	0.197	0.144	8.18
59	0.00795	0.00586	93	0.221	0.162	8.47
60	0.00872	0.00643	94	0.250	0.183	8.78
61	0.00956	0.00705	95	0.283	0.207	9.12
62	0.0105	0.00772	96	0.322	0.235	9.48
63	0.0115	0.00846	97	0.367	0.267	9.88
			98	0.421	0.306	10.32
			99	0.485	0.352	10.81
			100	0.562	0.407	11.34

refer to the gamma-energy, which is also the nuclear transition energy, rather than to the energy of the conversion electrons, we will use (32) instead of (31). The two expressions are simply connected by means of the relation $k = E - \gamma$.

The coefficients A and B have been computed for the range $Z = 30 - 100$ and can be found in Table 1. By means of this table a very accurate and simple determination of the conversion coefficients β_1^K can be made for any energy within the range of validity of (32). In order to give the reader an idea of the applicability of (32), conversion coefficients derived from it and Table 1 are compared in Fig. 1 with values of $k^3 \beta_1^K$ obtained by Rose. The values obtained by Rose have been complemented with a few exact calculations carried out by the present authors using the convergent expansions derived by one of them (Olsson 1959 b). For the heavy nuclei, the agreement is good, even for energy values far beyond the domain of convergence, but for such values it is considered

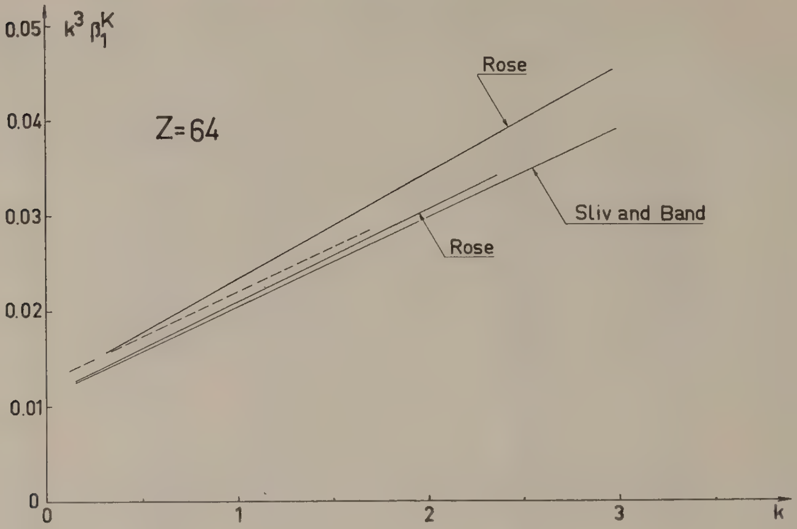

Fig. 1 a.

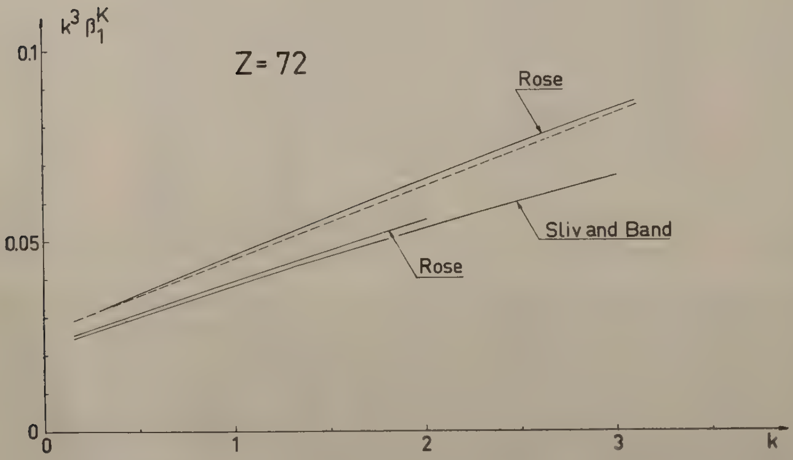

Fig. 1 b.

Fig. 1. The magnetic dipole conversion coefficients for the K-shell, multiplied by k^3 and plotted as functions of k, for four different Z-values. The dotted lines are obtained from formula (32) in the text and should coincide with the point nucleus values computed by Rose. The two lower curves in each plot give the finite nuclear size values obtained by Sliv and Band and by Rose.

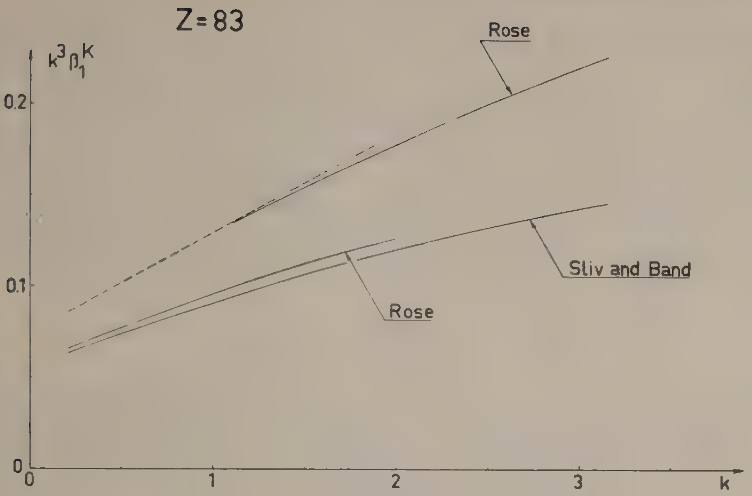
Z=83
$k^3\beta_1^K$
0.2
0.1
0
0
1
2
3
k
Rose
Sliv and Band
Rose

Fig. 1 c.

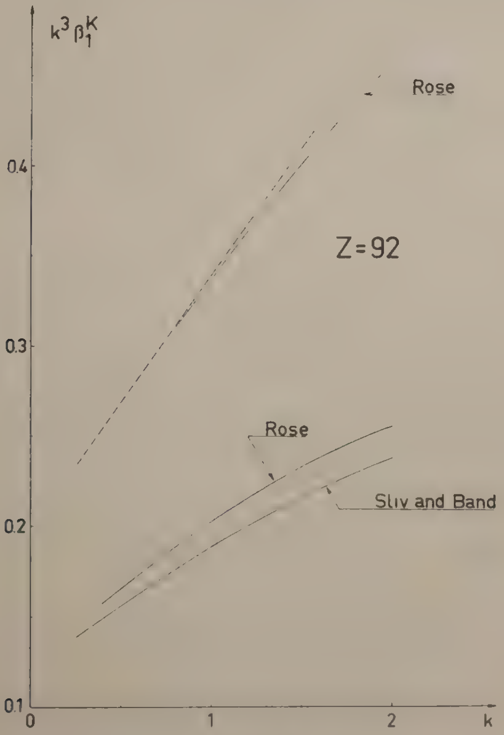
$k^3\beta_1^K$
Rose
0.4
Z=92
0.3
Rose
Sliv and Band
0.2
0.1
0
1
2
k

Fig. 1 d.

merely accidental and no attempt has been made to compute further coefficients in (31).

Formula (32) gives the unscreened point-nucleus conversion coefficients but, since we are now convinced of its practical applicability, we can correct the coefficients A and B for various effects having an influence on the internal conversion process. The fact that both the constants refer to threshold energies, for which considerable simplifications in the mathematical formalism are possible, should make the investigation of such corrections as simple as possible. We will next consider corrections due to the finite nuclear size. Correction terms will be added to the electron wave function and, from them, the finite nuclear size wave functions in the vicinity of the nucleus can be obtained.

Effects due to the finite size of the nucleus

The wave functions given in (2) are valid without modification for the bound states as well as the continuum states. Furthermore two independent solutions are obtained from (2) simply by choosing for γ either $\gamma = +\sqrt{\varkappa^2 - \alpha^2 Z^2}$ or $\gamma = -\sqrt{\varkappa^2 - \alpha^2 Z^2}$. In order to obtain values for the wave functions in the vicinity of the origin, we can use the power series expansion (Olsson 1959 a)

$$\Phi(\gamma, E, t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n \tag{33}$$

where the coefficient can be derived from the recurrence relation

$$(n + 2\gamma)\, c_{n+1} + E\, c_n + \frac{p^2}{4\alpha^2 Z^2}\, n\, c_{n-1} = 0$$

$$c_0 = 1, \quad c_{-n} = 0$$

We shall first derive the wave functions for bound states. Since the presence of the nucleus no longer makes it necessary to require $\chi_1(0) = \chi_2(0) = 0$, the wave functions will contain both the regular and irregular solutions of (3) but they are still submitted to the condition of normalizability. Normalization is possible only if the solutions decrease exponentially at infinity. The correct mixing of the regular and irregular solutions can be obtained from the well-known asymptotic behaviour of the wave functions and the result, which contains four $\Phi(\gamma, E, t)$ functions, can be conveniently expressed in terms of Whittaker functions.

$$\begin{aligned} \chi_1 &= \alpha Z \left[W_{n+\gamma, \gamma - \frac{1}{2}}(x) - \frac{n}{n+\gamma} \frac{E(\varkappa - \gamma)}{E\varkappa - \gamma} W_{n+\gamma, \gamma + \frac{1}{2}}(x) \right] \\ \chi_2 &= (\varkappa - \gamma) \left[W_{n+\gamma, \gamma - \frac{1}{2}}(x) - \frac{n}{n+\gamma} \frac{E(\varkappa + \gamma)}{E\varkappa - \gamma} W_{n+\gamma, \gamma + \frac{1}{2}}(x) \right] \end{aligned} \tag{34}$$

where $$x = 2\sqrt{1 - E^2}\, r, \quad n = \frac{E \alpha Z}{\sqrt{1 - E^2}} - \gamma.$$

The two Whittaker functions occurring here are defined as certain linear combinations of confluent hypergeometric functions:

$$\begin{aligned} W_{n+\gamma,\gamma-\frac{1}{2}}(x) &= \frac{\Gamma(1-2\gamma)}{\Gamma(1-n-2\gamma)}\, x^{\gamma} e^{-x/2}\, {}_1F_1(-n;\, 2\gamma;\, x) + \\ &\quad + \frac{\Gamma(2\gamma-1)}{\Gamma(-n)}\, x^{-\gamma+1} e^{-x/2}\, {}_1F_1(1-n-2\gamma;\, 2-2\gamma;\, x) \\ W_{n+\gamma,\gamma+\frac{1}{2}}(x) &= \frac{\Gamma(-1-2\gamma)}{\Gamma(-n-2\gamma)}\, x^{\gamma+1} e^{-x/2}\, {}_1F_1(1-n;\, 2\gamma+2;\, x) + \\ &\quad + \frac{\Gamma(1+2\gamma)}{\Gamma(1-n)}\, x^{-\gamma} e^{-x/2}\, {}_1F_1(-n-2\gamma;\, -2\gamma;\, x). \end{aligned} \tag{35}$$

If the parameter n is a positive integer or zero, the irregular terms in (35) are zero and both the components χ_1 and χ_2 vanish for $r=0$. This gives the ordinary hydrogen-like wave functions. In the presence of a nucleus of finite dimensions, the exterior solution (34) must be joined continuously to an interior solution at some point outside the nuclear charge distribution. This can be accomplished by changing the parameter n which is then no longer an integer. Due to the small dimensions of the nucleus, the deviation of n from an integral value is very small and sufficiently accurate wave functions can be obtained by considering only terms of first order in the deviations δn. Since solutions that are regular at the origin can be obtained for integral n-values only, it follows that the terms of first order in δn are singular at the origin and thus considerable changes in the electron densities can result in spite of the small value of δn but only where the terms containing $r^{-\gamma}$ are large, i.e. for $r \ll 1$.

Where $r \gg 1$, the corrections are of order δn and are thus negligible. Since we are here concerned with K-shell wave functions only, obtained from (34) if we put $n=0$, $E=\gamma$ and $\varkappa=-1$, we shall investigate (34) when n differs by a small amount δn from zero. We obtain, retaining first order terms only,

$$\begin{aligned} \chi_1 &= \alpha Z \left[W_{\delta n+\gamma,\gamma-\frac{1}{2}}(x) - \delta n \frac{1+\gamma}{2\gamma} W_{\gamma,\gamma+\frac{1}{2}}(x) \right] \\ \chi_2 &= -(1+\gamma) \left[W_{\delta n+\gamma,\gamma-\frac{1}{2}}(x) - \delta n \frac{1-\gamma}{2\gamma} W_{\gamma,\gamma+\frac{1}{2}}(x) \right]. \end{aligned} \tag{36}$$

From these expansions it is immediately seen that the small component, χ_1, is much more sensitive to the presence of nuclear charges than is the large component. We are only interested in the terms in $W_{\gamma,\gamma+\frac{1}{2}}(x)$ which can be large and thus we put

$$W_{\gamma,\gamma+\frac{1}{2}}(x) \approx \frac{\Gamma(1+2\gamma)}{x^{\gamma}} e^{x/2}.$$

This is exactly equal to the second term in $W_{\gamma,\gamma-\frac{1}{2}}(x)$, as can be seen from (35) since ${}_1F_1(-2\gamma;\, -2\gamma;\, x) = e^{x}$. Similarly we get

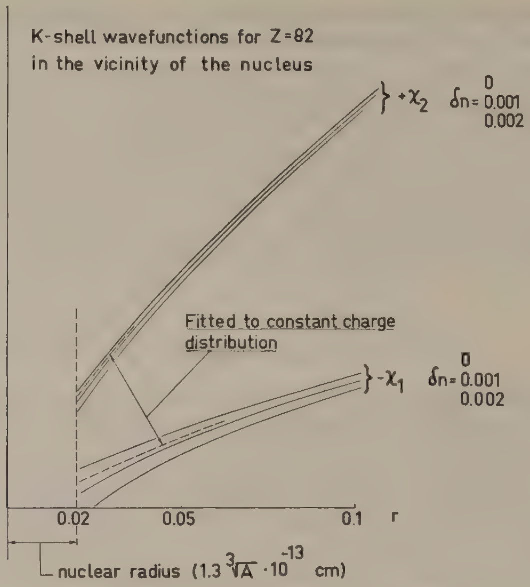

Fig. 2. Changes in the small and large components in the electron wave function near the nucleus due to variation of δn. The functions are obtained from formula (37) in the text. The small component is plotted, for convenience, with an opposite sign.

$$W_{\delta n+\gamma,\gamma-\frac{1}{2}}(x) \approx x^{\gamma} e^{-x/2} - \frac{\delta n\, \Gamma(2\gamma-1)\, e^{x/2}}{x^{\gamma-1}}\, {}_1F_1(1\,;\, 2-2\gamma\,;\, x).$$

For the sake of simplicity we write

$$W_{\delta n+\gamma,\gamma-\frac{1}{2}}(x) \approx x^{\gamma} e^{-x/2} - \frac{\delta n\, \Gamma(2\gamma-1)}{x^{\gamma-1}}$$

which finally gives the K-shell wave functions

$$\begin{aligned} N_i\chi_1 &= -\sqrt{\frac{\alpha Z(1-\gamma)}{\Gamma(1+2\gamma)}}\left[t^{\gamma} e^{-t/2} - \frac{\Gamma(2\gamma)\,\delta n}{t^{\gamma}}\left((1+\gamma)\,e^{t/2} + \frac{t}{2\gamma-1}\right)\right] \\ N_i\chi_2 &= \sqrt{\frac{\alpha Z(1+\gamma)}{\Gamma(1+2\gamma)}}\left[t^{\gamma} e^{-t/2} - \frac{\Gamma(2\gamma)\,\delta n}{t^{\gamma}}\left((1-\gamma)\,e^{t/2} + \frac{t}{2\gamma-1}\right)\right]. \end{aligned} \tag{37}$$

The corrected wave functions for reasonable values of δn are given in fig. 2. It should be mentioned that the change in the normalization constant is of order δn and can thus be neglected. The value of δn is obtained by joining the solutions (37) to interior solutions at some properly chosen point x_0 which may be taken as the nuclear radius. The value of the quotient of the interior solutions,

$$\left(\frac{\chi_1(x_0)}{\chi_2(x_0)}\right)^{\text{interior}},$$

depends on the nuclear charge distribution but it is very insensitive to variations in the energy due to the small value of the nuclear radius. This quotient determines δn which is obtained from

$$\left(\frac{\chi_1(x_0)}{\chi_2(x_0)}\right)^{\text{interior}} = -\frac{\alpha Z}{1+\gamma}\,\frac{1-\Gamma(2\gamma)\dfrac{\delta n}{x_0^{2\gamma}}\left[(1+\gamma)e^{x_0}+\dfrac{x_0}{2\gamma-1}\right]}{1-\Gamma(2\gamma)\dfrac{\delta n}{x_0^{2\gamma}}\left[(1-\gamma)e^{x_0}+\dfrac{x_0}{2\gamma-1}\right]}. \tag{38}$$

Since $\left(\dfrac{\chi_1}{\chi_2}\right)^{\text{interior}}$ is negative for the small values of x_0 considered here, we must have

$$1-\Gamma(2\gamma)\frac{\delta n}{x_0^{2\gamma}}\left[(1+\gamma)e^{x_0}+\frac{x_0}{2\gamma-1}\right]>0$$

or

$$\delta n<\frac{x_0^{2\gamma}}{\Gamma(2\gamma)\left[(1+\gamma)e^{x_0}+\dfrac{x_0}{2\gamma-1}\right]}$$

which gives $\delta n<0.0016$ even for the heaviest nuclei.

The continuum wave functions will now be treated in a similar manner. We shall use χ^+ to denote the solutions (2) when γ is positive and χ^- the solutions when γ is negative. The normalization constant for the χ^- wave functions is obtained from (24) by changing the sign of γ.

The normalized free wave functions with an arbitrary asymptotic phase can be written

$$\begin{aligned}\chi_1&=\frac{N_f^+}{A}\chi_1^+ +\delta n_f\frac{N_f^-}{A}\chi_1^-\\ \chi_2&=\frac{N_f^+}{A}\chi_2^+ +\delta n_f\frac{N_f^-}{A}\chi_2^-\end{aligned} \tag{39}$$

where A and δn_f are two parameters. A depends on δn_f since only one parameter remains in the general solution if the solutions are normalized. The normalization constants $N_f^\pm$, chosen as in (24), give the following asymptotic expressions

$$\begin{aligned}\chi_1^\pm&\approx\sqrt{\frac{E-1}{\pi p}}\sin\left[pr+\eta\ln 2pr\mp\frac{\pi\gamma}{2}-\arg\Gamma(\pm\gamma+i\eta)+\delta^\pm\right]\\ \chi_2^\pm&\approx-\sqrt{\frac{E+1}{\pi p}}\cos\left[pr+\eta\ln 2pr\mp\frac{\pi\gamma}{2}-\arg\Gamma(\pm\gamma+i\eta)+\delta^\pm\right]\end{aligned} \tag{40}$$

where
$$\tan\delta^\pm=\frac{\alpha Z(E-1)}{p(\varkappa\mp\gamma)}.$$

Generally we have

$$\frac{1}{A}\left[\sin\left(x+\delta_1\right)+\delta n_f \sin\left(x+\delta_2\right)\right]=\sin\left(x+\delta_3\right)$$

$$\frac{1}{A}\left[\cos\left(x+\delta_1\right)+\delta n_f \cos\left(x+\delta_2\right)\right]=\cos\left(x+\delta_3\right)$$

if

$$\tan\delta_3=\frac{\sin\delta_1+\delta n_f \sin\delta_2}{\cos\delta_1+\delta n_f \cos\delta_2}$$

and

$$A^2=1+2\,\delta n_f \cos\left(\delta_1-\delta_2\right)+\left(\delta n_f\right)^2.$$

The solution (39) thus has the asymptotic form (40) but whith a different phase. Further it is seen that the changes in the normalization constants are of the order δn_f which, for small kinetic energies of the free electron, is of the same order as δn, introduced into the wave functions for the bound states.

Therefore, only a few terms in the expansions of χ^- are needed and these terms can be obtained from (33). We obtain

$$\chi_1^-=\frac{\alpha Z}{t^\gamma}\left[1+\frac{E t}{2\gamma}+\frac{E\varkappa-\gamma}{\varkappa-\gamma}\,\frac{t}{(-2\gamma)_2}+\cdots\right]$$

$$\chi_2^-=\frac{(\varkappa+\gamma)}{t^\gamma}\left[1+\frac{E t}{2\gamma}+\frac{E\varkappa-\gamma}{\varkappa+\gamma}\,\frac{t}{(-2\gamma)_2}+\cdots\right]$$

and, for the free electrons, we thus obtain

$$\begin{aligned}\chi_1&=N_f^+\,\chi_1^+ +\delta n_f N_f^-\,\frac{\alpha Z}{t^\gamma}\left[1+\left(\frac{E}{2\gamma}+\frac{E\varkappa-\gamma}{\varkappa-\gamma}\,\frac{1}{(-2\gamma)_2}\right)t+\cdots\right]\\ \chi_2&=N_f^+\,\chi_2^+ +\delta n_f N_f^-\,\frac{(\varkappa+\gamma)}{t^\gamma}\left[1+\left(\frac{E}{2\gamma}+\frac{E\varkappa-\gamma}{\varkappa+\gamma}\,\frac{1}{(-2\gamma)_2}\right)t+\cdots\right].\end{aligned}\tag{41}$$

The parameter δn_f is obtained by connecting the solutions (41) with the interior solutions as in (38).

For an $s_{\frac{1}{2}}$-state, for which $\varkappa=-1$, we have

$$\begin{aligned}\chi_1&=N_f^+\,\chi_1^+ +\delta n_f N_f^-\,\frac{\alpha Z}{t^\gamma}\left[1+\frac{t}{2\gamma}\left(E+\frac{E+\gamma}{(1+\gamma)(2\gamma-1)}\right)+\cdots\right]\\ \chi_2&=N_f^+\,\chi_2^+ -\delta n_f N_f^-\,\frac{(1-\gamma)}{t^\gamma}\left[1+\frac{t}{2\gamma}\left(E+\frac{E+\gamma}{(1-\gamma)(2\gamma-1)}\right)+\cdots\right].\end{aligned}\tag{42}$$

The correction terms should, of course, be used only for $t<1$.

The most important corrections due to the finite nuclear size come, however, from the effects inside the nucleus where the rather strong singularities of the fields (which nearly make the integrals non-existing (¹)) considerably ex-

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(¹) In the limit of a point nucleus the integrals may actually diverge if the limit is not carefully derived. This circumstance has given rise to the so-called gauge-problem, the spurious nature of which was first pointed out by Kramer (1956, 1957).

aggerate the interaction between the shell electrons and the nucleus. If the nucleus is extended to finite dimensions, both the photon-field and the electron wave functions are finite everywhere and the interaction between the nucleons and the electrons is more correctly estimated when the latter are inside the nucleus. To neglect the interaction completely in this region (Sliv) is, physically, a more reasonable approximation than to consider the nucleus as a point. However, in transitions where the integrand of the matrix element is not singular, or only weakly singular, the point-nucleus approximation is good. The latter statement holds for the majority of nuclear transitions. An important improvement of the theory of internal conversion was thus made by Sliv who removed the incorrectly retained singularities from the theory. That this removal of the singularities is the essential effect will become clear from the present investigation. Indications in the same direction are also evident from the calculations of Rose (1958). This author used a model where the radiation sources were concentrated to the origin (no-penetration model). As a result he had the strong near-zone radiation field inside and in the vicinity of the nucleus. In this way, the strength of the interaction near the nucleus will be exaggerated but, nevertheless, only small differences from the numerical results of Sliv and Band were obtained. The reason for this is the use of finite nuclear size electron wave functions which removes the singularities in the matrix elements. Thus, for gamma-transitions that proceed at a normal rate, the conversion coefficients are insensitive to the nuclear structure. The differences betwen the calculations of Sliv and Band and those of Rose indicate how large a dependence on the nuclear structure that can be expected.

A detailed analysis of the effects referred to above can be made by investigating their influence on the parameters A and B in formula (32). For the heavy nuclei, for which the applicability of (32) extends well into the energy region of physical interest, sufficiently accurate results can be obtained for any nuclear model.[1] We will not investigate here the influence on the internal conversion coefficients that can be expected from the nuclear structure effects. In order to do this for a definite nuclear transition a particular nuclear model and the nuclear wave functions for that particular model and nuclear transition are required. Such investigations are only motivated for very slow gamma-transitions which are many orders of magnitude slower than the single particle estimates. Not knowing in detail what to put into the nucleus we shall thus use an empty nucleus in the same way as Sliv and Band. When necessary, the empty nucleus can be replaced by suitable expressions which take account of the nuclear structure and the conversion coefficients can be corrected for the effects of nuclear structure in a way similar to that of Church and Weneser (1956).

We will therefore not integrate over the nucleus and the corrected matrix element can be written

$$R^c_{\varkappa'}(L) = -i(-1)^L N_i N_f \int_{r_0}^{\infty} j_{-L}(kr)\,[\chi_1^i \chi_2^f + \chi_1^f \chi_2^i]\,dr, \tag{43}$$

where the electron wave functions are now given in (36) and (41). In order to obtain an expression simple enough for the correction term in $R_{\varkappa'}(L)$ we retain only the leading term in the corrections added to the wave functions.

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[1] Strong deviations from the spherical shape may form exceptions.

$$k^2 R^c_{-1}(1) = k^2 R_{-1}(1) + i N_f^+ \sqrt{\frac{\alpha Z(1-\gamma)}{\Gamma(1+2\gamma)}}\, 2(1+\gamma) \int\limits_0^{r_0} \frac{(2\alpha Z r)^{2\gamma}}{r^2}\, dr +$$

$$+ 2 i N_f^+ \sqrt{\frac{\alpha Z(1-\gamma)}{\Gamma(1+2\gamma)}} \left[(1+\gamma)\Gamma(2\gamma)\,\delta n - \frac{N_f^-}{N_f^+}\,\delta n_f\right] \int\limits_{r_0}^{1} \frac{dr}{r^2} =$$

$$= k^2 R_{-1}(1) + i N_f^+ 2(1+\gamma) \sqrt{\frac{\alpha Z(1-\gamma)}{\Gamma(1+2\gamma)}} \left\{\frac{(2\alpha Z)^{2\gamma} r_0^{2\gamma-1}}{2\gamma-1} + \right.$$

$$\left. + \left[\Gamma(2\gamma)\,\delta n - \frac{\delta n_f}{1+\gamma}\frac{N_f^-}{N_f^+}\right]\frac{1}{r_0}\right\}. \qquad (44)$$

The first term is the point-nucleus matrix element and the second term contains the corrections due to the finite nuclear size. The term containing $r_0^{2\gamma-1}$ is the most important one and removes the unduly strong interaction connected with the point-nucleus model. The terms containing the parameters δn and δn_f represent the static corrections which arise from effects due to the changed electron densities near the nucleus. The nuclear radius, r_0, is measured in units of $\hbar/mc$. The two parameters δn and δn_f are not independent but connected through the approximate relation

$$\left(\frac{\chi_1}{\chi_2}\right)_{r_0}^{\text{final state}} \approx \left(\frac{\chi_1}{\chi_2}\right)_{r_0}^{\text{initial state}}, \qquad (45)$$

since the two ratios have to be equal to the ratio of the interior solutions which is insensitive to changes in the energy. From (45) it follows that

$$\frac{N_f^-}{N_f^+}\,\delta n_f \approx -(1+\gamma)\Gamma(2\gamma)\,\delta n. \qquad (46)$$

Writing (44) as
$$k^2 R^c_{-1}(1) = k^2 R_{-1}(1) + k^2 \Delta R_{-1}(1) \qquad (47)$$

and using (46) we obtain

$$k^2 \Delta R_{-1}(1) = i N_f^+ \sqrt{\frac{\alpha Z(1-\gamma)}{\Gamma(1+2\gamma)}}\, 4\alpha Z(1+\gamma)(2\alpha Z r_0)^{2\gamma-1}\left[\frac{1}{2\gamma-1} + \frac{2\Gamma(2\gamma)\,\delta n}{(2\alpha Z r_0)}\right]. \qquad (48)$$

The correction term is energy-dependent only in the factor N_f which here is

$$N_f^+ = \frac{1}{\sqrt{\alpha Z}\,\Gamma(1+2\gamma)}\left(1 + \frac{\gamma}{3}p^2 + \cdots\right).$$

Then
$$\left[\frac{\partial}{\partial E} k^2 \Delta R_{-1}(1)\right]_{E=1} = \frac{2\gamma}{3}\,[k^2 \Delta R_{-1}(1)]_{E=1}. \qquad (49)$$

We now write the internal conversion coefficients, corrected for the finite nuclear size, in the same way as that used in (31)

$$k^3 \beta_1^K = \frac{4\pi\alpha}{3} |k^2 R^c|^2 =$$

$$= \frac{4\pi\alpha}{3} [|k^2 R| - |k^2 \Delta R|]^2 =$$

$$= A_E^c + (E-1) B^c + \cdots = A^c + k B^c + \cdots$$

where

$$A^c = A_E^c - B^c (1-\gamma). \tag{50}$$

For simplicity, we have dropped the indices for R. We obtain

$$A_E^c = A_E \left(1 - \left|\frac{\Delta R}{R}\right|\right)^2_{E=1} \tag{51}$$

and finally

$$B^c = \left(1 - \left|\frac{\Delta R}{R}\right|\right) \left[B - \frac{4 A_E \gamma}{3} \left|\frac{\Delta R}{R}\right|\right]. \tag{52}$$

The corrections to be made in A and B thus can be expressed in terms of the ratio $\Delta R/R$ for which the general expression is

$$\left|\frac{\Delta R}{R}\right| = \frac{4\alpha Z (1+\gamma)(2\alpha Z r_0)^{2\gamma-1}}{T(2\gamma-1)} \left[1 + 2(2\gamma-1) \frac{\Gamma(2\gamma)\,\delta n}{(2\alpha Z r_0)^{2\gamma}}\right] \tag{53}$$

where T is given in table 1. The general expression is

$$T = 2(1+\gamma) k_0^2 J_1(1, 1) + \frac{2}{(2\gamma)_2} k_0^2 J_2(1, 1). \tag{54}$$

In (53) we observe that the second term contains the factor $\Gamma(2\gamma)\,\delta n/(2\alpha Z r_0)^{2\gamma}$ which is a quantity that can be determined from the eigenvalue equation (38). The approximate value of δn was discussed in connection with (38) and comparing with (53) we can easily see that the second term in (53) is rather small compared with the first term.

Magnetic dipole conversion coefficients with finite nuclear size corrections

In this section, we shall apply the results of the previous sections to magnetic dipole K-shell conversion in lead. From Table 1 we obtain, for the point-nucleus, the conversion coefficient as function of the nuclear transition energy k:

$$\beta_1^K = \frac{1}{k^3}(0.0675 + 0.0497\,k). \tag{55}$$

This simple formula gives the β_1^K obtained by Rose with an error less than one per cent for transition energies up to 600 keV. Further we have

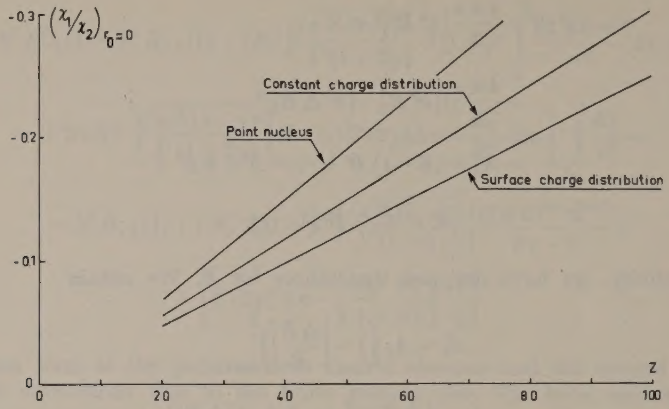

Fig. 3. The ratio at the nuclear surface between the small and the large components of the interior electron wave function. This ratio is very insensitive to the choice of the nuclear radius and to the changes in the energy. The ratios plotted here are computed for $r_0 = 0$ and $E = 1$.

$$\left|\frac{\Delta R}{R}\right| = \frac{2.394 \times 1.801\,(1.197\,r_0)^{0.602}}{6.12 \times 0.602}\,[1 + 2 \times 0.602\,x] \tag{56}$$

where x can be obtained from the charge distribution within the lead nucleus, since

$$x = \frac{\Gamma(2\gamma)\,\delta n}{(2\alpha Z r_0)^{2\gamma}}.$$

The quantity x can be obtained from formula (38) and the value of χ_1/χ_2 from Fig. 3. For $Z = 82$, we obtain $x \approx 0.15$ and then the correction due to the change of the electron density outside the nucleus will be about 18 % of the correction due to the removal of the unduly strong fields inside the nucleus. Fig. 4 shows the conversion coefficient for two different values of the nuclear radius, for a constant charge distribution within the nucleus, and for the nuclear charge distributed over the nuclear surface. The results are in good agreement with the values obtained by Sliv and Band. The small differences are due to the use of the approximate formula (53) and the neglection of screening in the present work. Screening is thus, at least for small energies, unimportant for magnetic dipole conversion in the K-shell.

Since both the coefficients A_E and B, defined in (31), are insensitive to the value of the binding energy of the converted electron, the coefficient A, defined by $A = A_E - B(1-\gamma)$, can easily be corrected for the true binding energy E_B. We then have

$$A = A_E - B\,E_B.$$

This increases the values of the conversion coefficients for the heavy nuclei by one or two per cent. For lead the values of Fig. 4 should be increased by about one per cent. The error in (53) is, however, perhaps two per cent since only the

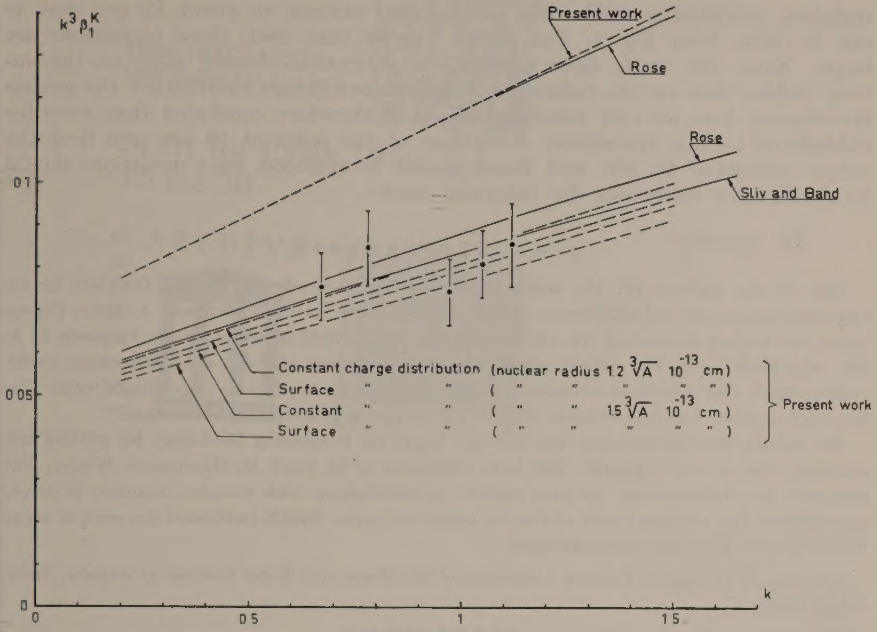

Fig. 4. Magnetic dipole conversion coefficients for conversion in the K-shell, computed for different nuclear charge distributions and for different values of nuclear radius. The dotted lines represent results obtained by the present authors. The curve valid for the smaller nuclear radius and for constant charge distribution should agree well with the values obtained by Sliv and Band since the essential difference between the two calculations is that the latter authors have taken screening into account. Screening should not be of importance in this case since the transition matrix elements are only sensitive to the electron wave functions near the nucleus. The two upper curves represent point-nucleus values.

leading term in the expansions of the wave functions has been retained, in order to obtain a simple expression. From Fig. 4 it is clear that distinctions between possible values for the nuclear radius and reasonable nuclear charge distributions require an experimental accuracy of a few per cent. A few careful absolute measurements of $M1$ conversion coefficients are shown in Fig. 4 (Stockendal and Hultberg, 1959). It should perhaps be pointed out that these measurements do not represent the best accuracy possible with the method of Hultberg and Stockendal (Hultberg and Stockendal, 1959). In favourable cases an accuracy of better than 5% may be reached. As far as these experimentally studied transitions can be regarded to have a pure $M1$ character the preference for the finite nuclear size values can be seen. To within a few per cent, the latter are obtained by removing the singularity at the origin which, as has been stressed previously, greatly exaggerates the interaction between shell electrons and nucleons when the electrons are in the vicinity of the nucleus. The remaining corrections that depend on screening, the nuclear charge distribution, deviations from the Coulomb potential near the nucleus due to non-spherical shape of the nucleus and the radiation mechanism inside the nucleus can amount to about 12 per cent as can be seen from Fig. 4. For nuclei, heavier than lead, these corrections are larger. Since the static finite nuclear size corrections depend more on the nuclear radius than on the variations of the nuclear charge distribution, the models investigated here are not unreasonable. It is therefore concluded that, even for unhindered nuclear transitions, deviations of the order of 10 per cent from the values calculated by Sliv and Band cannot be excluded. Such deviations should be particularly interesting for deformed nuclei.

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REFERENCES

E. CHURCH and J. WENESER, Phys Rev. *104*, 1382 (1956).
A. ERDÉLYI, Math. Z. *40*, 693 (1936).
——, Proc. Roy. Soc. Edinburgh A *62*, 379 (1948).
T. GREEN and M. E. ROSE, Phys. Rev. *110* 105 (1958).
S. HULTBERG, Arkiv för Fysik *15*, 307 (1959).
S. HULTBERG and R. STOCKENDAL, Arkiv för Fysik *14*, 36 (1959).
G. KRAMER, Z. f. Physik *146*, 187 (1956).
——, Z. f. Physik *147*, 628 (1957).
S. G. NILSSON and J. O. RASMUSSEN, Nuclear Physics *5*, 617 (1958).
P. O. M. OLSSON, Arkiv för Fysik *15*, 131 (1959 a).
——, Arkiv för Fysik *15*, 159 (1959 b).
——, Arkiv för Fysik *15*, 289 (1959 c).
A. S. REINER, Nuclear Physics *5*, 544 (1958).
M. E. ROSE, G. GOERTZEL, B. SPINRAD, J. HARR, and P. STRONG, Phys. Rev. *83*, 79 (1951).
M. E. ROSE, Multipole Fields, John Wiley & Sons, New York (1955).
M. E. ROSE, Internal conversion coefficients, North-Holland Publishing Company (1958).
L. A. SLIV, J. Exp. Theor. Phys. *21*, 770 (1951).
L. A. SLIV and M. LISTENGARTEN, J. Exp. Theor. Phys. *29*, 29 (1952).
L. A. SLIV and J. BAND, Tables of internal conversion coefficients of gamma-rays, published by the Academy of the USSR, Moscow–Leningrad 1956.
——, Coefficients of internal conversion of gamma radiation. Part 2: L-shell. Published by the Academy of Sciences of the USSR, Moscow-Leningrad 1958.
R. STOCKENDAL and S. HULTBERG, Arkiv för Fysik *15*, 33 (1959).
F. TRICOMI, An. Mat. Pura Appl. *IV 26*, 141 (1949). See also A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER and F. TRICOMI, Higher transcendental functions, vol. 1, McGraw-Hill Book Co., New York 1953.

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